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VACUUM FLUCTUATION EFFECTS IN CIRCUITS AND ELECTRON STREAMS

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Abstract

Field quantization is applied to an electrical oscillating circuit. Damping effects are treated by perturbation theory. Quantum effects occur both in the damping and in the noise, and are discussed in detail. An interpretation is given of the infinite zero point contribution which appears in the theory of Callen and Welton. The average electromagnetic field energy of an oscillator with capacitance C, conductance G, and natural frequency ω, as a function of time is given by

$$U = \left[\frac{1}{2}\hbar\omega + \frac{\hbar\omega}{\exp(\hbar\omega/kT)-1}\right]\left[1 - e^{-Gt/C}\right] + U_0 e^{-Gt/C} .$$

The mean squared noise voltage which would be measured in an experiment with a damped oscillator is given by

$$\overline{V^2} = \frac{1}{C}\left[\frac{1}{2}\hbar\omega + \frac{\hbar\omega}{\exp(\hbar\omega/kT)-1}\right] .$$

The maximum noise power which a conductance G at temperature T can transfer to a damped oscillator approaches the value

$$\frac{G\hbar\omega}{C\left[\exp(\hbar\omega/kT)-1\right]} .$$

These results are extended to include interaction of the circuit with radiation fields.

The vacuum fluctuations are shown to be observable in electromotive force measurements and in electron stream noise.

Introduction

During the past quarter of a century a number of theoreticians have turned their attention to the problem of constructing a satisfactory quantum theory of the electromagnetic field. The problem is a very difficult one, because it is intimately tied up with the structure of the elementary particles, about which almost nothing is known. For a large number of calculations the theory gave infinite results. For a long time it was not understood whether the infinite results were a consequence of faults in the theory or a lack of knowledge of the elementary particles. Further difficulty lay in the scarcity of experimental data on the order of magnitude of some of the effects for which calculations gave infinite results.

One of the effects for which calculations gave an infinite result was the shift of the energy levels of an electron in a hydrogen atom due to interaction with the radiation fields. This shift was measured with precision in 1947. Bethe and Weisskopf have shown that the theory correctly predicted the magnitude of the shift if we handle the interaction of the electron with the radiation field in the following way. The infinite result for the bound electron in a hydrogen atom is compared with the infinite result for a free electron. The difference does not diverge and predicts closely the correct value for the energy level shift. Subsequently, T. A. Welton showed that the shift could be calculated if we assume that each mode of the electromagnetic field has zero point fluctuations corresponding to a zero point energy $1/2\ \hbar\omega$. We then calculate classically the interaction of the electron with the field fluctuations.

Ascribing a zero point energy $1/2\,\hbar\omega$ to each mode of the electromagnetic field leads to an infinite zero point energy for the vacuum. It can be shown, however, that the zero point fluctuations of the fields are entirely independent of the choice of zero point energy. They are, rather, a consequence of field quantization and are due to the random interactions between the vacuum and the apparatus for measuring the fields. The zero point field fluctuations have been shown to produce observable effects on the energy levels of atomic hydrogen. The question considered in this investigation is whether these fluctuations also give rise to observable effects in electric circuits and in electron stream noise. If these fluctuations do in fact have observable effects in electric circuits, then they set an ultimate limit to the precision of ordinary electrical measurements and to the sensitivity of electronic devices in general. This is a limit which would be approached as the quantity $\frac{\hbar\omega}{kT}$ becomes large (i.e. low temperature or high frequency).

To investigate this point, the simplest circuit and electron beam problems are investigated here, using the quantum theory of fields and the quantum statistics.

Section I

Quantum Theory of a Damped Electrical Oscillator and Noise*

Virtually all of the phenomena occurring in electric circuits are described classically in a satisfactory way by Maxwell's equations Application of classical statistics has led to a satisfactory understanding of most electrical fluctuation phenomena. The classical description is usually adequate because ordinary measurements are made at room temperature with circuit currents exceeding noise currents. If measurements were made at low temperatures, with smaller currents, the quantum effects would be significant. No experiments have been carried out under such conditions. The present paper is an effort to provide some basic work for a general quantum theory of circuits and noise.

Recently Callen and Welton[1] presented an elegant quantum theory of noise. Their results showed as one of the quantum effects an infinite zero-point noise contribution for a pure resistance. The theory to be presented here gives insight into the origin of the infinite zero-point contribution and predicts finite quantum effects in certain experiments.

An Oscillating Circuit With No Dissipation

We consider first an electrical oscillator which we imagine made up of perfect conductors with no radiation. One is tempted to treat such a system as Fig. 1 as an ensemble of particles and to discuss its behavior in terms of charges and currents. This procedure leads to difficulties because with perfect conductors there are no tangential

* This section was published as it stands in the Physical Review Volume 90, 5, 977, June 1, 1953.

electric fields near the conductors. In order to allow currents to change without electric fields, charged particles without mass or an infinite number of carriers with mass would be required. To avoid these difficulties we choose to discuss the fields. The energy is

$$U = \frac{1}{8\pi}\int_V (E^2 + H^2)\,d\tau, \qquad (1)$$

where $\bar{E}$ and $\bar{H}$ are the electric and magnetic fields, and the integral is throughout space. We represent the magnetic vector potential as the product of a time-dependent and space-dependent part $q(t)\bar{A}(r)$. In terms of the vector potential,

$$\bar{E} = -(1/c)\dot{q}\bar{A}, \quad \bar{H} = q\nabla \times \bar{A} \qquad (2)$$

Fig. 1. Electrical oscillator with no dissipation.

From (2) we get, using Maxwell's equations,

$$\nabla \times \bar{H} = \left[\nabla(\nabla\cdot\bar{A}) - \nabla^2\bar{A}\right] q = -(1/c^2)\ddot{q}\bar{A} \qquad (3)$$

The fields are entirely outside of the perfectly conducting boundaries, and the most general[2] solution of Maxwell's equations can be expressed in terms of potentials such that the divergence of $\bar{A}$ is zero, and the scalar potential is also zero. If q oscillates harmonically with time, with angular frequency ω, (3) becomes

$$\nabla^2\bar{A} + (\omega^2\bar{A}/c^2) = 0 \qquad (4)$$

2. L. I. Schiff, <u>Quantum Mechanics</u> (McGraw-Hill Book Co., Inc., New York, 1949), p. 265

We normalize $\bar{A}$ so that

$$\int_v A^2 d\tau = 4\pi c^2.$$

We introduce a variable p canonically conjugate to q by letting $p = \mathring{q}$; from (2), $\bar{E} = -(1/c)p\bar{A}$. The total energy in the electric field becomes

$$\frac{1}{8\pi}\int_v E^2 d\tau = \frac{p^2}{8\pi c^2}\int A^2 d\tau = \frac{p^2}{2}. \qquad (5)$$

The magnetic energy is

$$\frac{1}{8\pi}\int_v H^2 d\tau = \frac{q^2}{8\pi}\int_v (\nabla \times \bar{A})^2 d\tau = \frac{q^2}{8\pi}\left[\int_s \bar{A} \times \nabla \times \bar{A} \circ d\bar{s} + \int_v \bar{A} \circ \left[\nabla(\nabla \cdot \bar{A}) - \nabla^2 \bar{A}\right] d\tau\right]. \qquad (6)$$

Substitution of (2) into the first term on the right side of (6) reduces it to

$$(qc/8\pi\omega)\int_s \bar{E} \times \bar{H} \circ d\bar{s},$$

where the surface integral is over a closed surface surrounding the circuit. From Poynting's theorem this term is proportional to the radiated power which is postulated to be zero.† Equation (6) becomes

$$\frac{1}{8\pi}\int_v H^2 d\tau = \frac{q^2\omega^2}{2}. \qquad (7)$$

The Hamiltonian (1) is now

$$H = \frac{1}{2}(p^2 + \omega^2 q^2). \qquad (8)$$

Our variables p and q must be noncommutable operators; otherwise it would be possible[3] to measure in the same region simultaneously the electric field and the magnetic field with arbitrarily great precision. We therefore adopt the commutation rule:

$$pq - qp = -i\hbar. \tag{9}$$

Expressions (8) and (9) make the problem of the undamped electrical oscillator formally identical with the harmonic oscillator. The wave functions for q are the well-known harmonic oscillator wave functions. The allowed values for the energy are $U = (n + \frac{1}{2})\hbar\omega$.

Harmonic Oscillator With Dissipation

We represent a damped oscillator by an oscillator of the type discussed above, coupled to a resistance, as shown in Fig. 2. We follow the general method of Callen and Welton.[1] The Hamiltonian can be written as

$$H = \frac{1}{2}(p^2 + \omega^2 q^2) + H_R + Q\int \bar{E} \cdot \overline{d\ell} \tag{10}$$

Here H_R is the Hamiltonian of the ensemble of particles making up the resistance; Q is a function of the coordinates and momenta of the particles of the resistance, and the line integral is over the length of the resistance. In terms of the capacity C we can write

$$\frac{1}{8\pi}\int_v E^2 d\tau = \frac{1}{2C}\left[\int \bar{E} \cdot \overline{d\ell}\right]^2 = \frac{1}{2}p^2. \tag{11}$$

Making use of (11) the Hamiltonian becomes

3. W. Heisenberg, Physical Principles of the Quantum Theory (Dover Publications, New York, 1950), p. 50.

$$H = \frac{1}{2}\left(p^2 + \omega^2 q^2\right) + H_R + \left(pQ/\sqrt{C}\right). \tag{12}$$

We treat the last term as an interaction term which will cause transitions with exchange of energy between the LC circuit and the resistance. We assume that the oscillator is weakly damped; if we use first order perturbation theory, the transition probability can be shown to be:

$$W_r = \frac{2\pi}{\hbar}\Big[\rho(E_r+\hbar\omega)\langle E_r|Q|E_r+\hbar\omega\rangle^2 \times \langle E_F|\tfrac{p}{\sqrt{C}}|E_F-\hbar\omega\rangle^2 + \rho(E_r-\hbar\omega) \times \langle E_r|Q|E_r-\hbar\omega\rangle^2 \langle E_F|\tfrac{p}{\sqrt{C}}|E_F+\hbar\omega\rangle^2\Big] \tag{13}$$

The symbol $\langle E_R|Q|E_R + \hbar\omega\rangle$ indicates the matrix element of the operator corresponding to Q between the quantum states of the resistance with eigenvalues E_R and $E_R + \hbar\omega$, $\langle E_F\,|\,p/\sqrt{C}\,|\,E_F - \hbar\omega\rangle$ has the corresponding meaning for the quantum states of the field, $\rho(E_r + \hbar\omega)$ is the density in energy of the quantum states of the resistance in the vicinity of the energy $E_R + \hbar\omega$. Expression (13) gives the total transition probability from an eigenstate of the unperturbed system in which the field has the eigenvalue E_F, and the resistance has the eigenvalue E_R. We may assume that initially the circuit is in an eigenstate (before being coupled to the resistance). There will never be enough information about the resistance to say that it is in an eigenstate, but its state will be partially specified in that its temperature will be known. It is therefore necessary to average (13) over an ensemble of similar systems, the result is

$$W_r = \frac{2\pi}{\hbar}\Big[\langle E_F|\frac{p}{\sqrt{C}}|E_F-\hbar\omega\rangle^2\int_0^\infty \rho(E_r+\hbar\omega)$$
$$\times\langle E_r|Q|E_r+\hbar\omega\rangle^2\rho(E_r)f(E_r)\,dE_r$$
$$+\langle E_F|\frac{p}{\sqrt{C}}|E_F+\hbar\omega\rangle^2\int_{\hbar\omega}^\infty \rho(E_r-\hbar\omega)$$
$$\times\langle E_r|Q|E_r-\hbar\omega\rangle^2\rho(E_r)f(E_r)\,dE_r \qquad (14)$$

Consider $f(E)$ the statistical weighting factor, and $f(E+\hbar\omega)/f(E) = \exp-(\hbar\omega/kT)$. The second integral has $\hbar\omega$ as a lower limit because energy is conserved in these transitions, and no resistance in the ensemble can undergo transitions which reduce its energy if its energy is less than $\hbar\omega$. We introduce the quantity

$$S = \frac{2\pi}{\hbar}\int_0^\infty \rho(E_r+\hbar\omega)\langle E_r|Q|E_r+\hbar\omega\rangle^2\rho(E_r)f(E_r)\,dE_r \qquad (15)$$

By making a change of variable in the second integral of (14) and making use of (15) we can put (14) into the form:

$$W_r = S\Big[\langle E_F|p/\sqrt{C}|E_F-\hbar\omega\rangle^2+\langle E_F|p/\sqrt{C}|E_F+\hbar\omega\rangle^2\exp-(\hbar\omega/kT)\Big]. \qquad (16)$$

The first term of (16) is the probability per unit time that the circuit and fields will lose a quantum, and the second term is the probability per unit time that the circuit and fields will gain a quantum. The net probability that a quantum will be lost will be the difference of the two terms. The average rate of change of energy of a circuit in the ensemble will be

$$dU/dt = S\hbar\omega\Big[\langle E_F|p/\sqrt{C}|E_F+\hbar\omega\rangle^2\exp-(\hbar\omega/kT)$$
$$-\langle E_F|p/\sqrt{C}|E_F-\hbar\omega\rangle^2\Big]. \qquad (17)$$

If we insert the well-known harmonic oscillator matrix elements into (17) the result is

$$\frac{dU}{dt} = \frac{S(\hbar\omega)^2}{2C}\left[(n+1)\exp - (\hbar\omega/kT) - n\right], \qquad (18)$$

where n is the quantum number of the circuit. Equation (18) states that the circuit may either gain or lose energy, depending upon n. It is interesting that the equilibrium value of n for which (18) is zero is

$$n = 1/\left[\exp(h\omega/kT) - 1\right].$$

The principal quantum effects are evident in (18); classically the rate of energy loss would be proportional to the energy at time t. This is only true in (18) if n is large.

If the relation for the energy, $U = (n + \frac{1}{2})\hbar\omega$, is inserted into (18) and the resulting equation integrated,† we obtain

$$U = \left[\frac{1}{2}\hbar\omega + \frac{\hbar\omega}{\exp(\hbar\omega/kT) - 1}\right]$$
$$\times\left[1 - \exp\left\{-\frac{\hbar\omega S}{2C}\left[1 - \exp - (\hbar\omega/kT)\right]t\right\}\right]$$
$$+ U_0 \exp\left\{-\frac{\hbar\omega S}{2C}\left[1 - \exp - (\hbar\omega/kT)\right]t\right\}. \qquad (19)$$

In Eq. (19), U is the average energy of a circuit in the ensemble as a function of time and the initial energy U_0; C is the capacity, and S is defined by (15). For large energy (classical limit) the second term is the only significant one. Comparing this with the known classica solution $U = U_0 e^{-Gt/C}$, where G is the conductance, we obtain:

† The procedure of this section is based on the discussion of E.C. Kemble, The Fundamental Principles of Quantum Mechanics (McGraw-Hill Book Co., Inc., New York, 1937), Chapter 12.

$$G = \frac{\hbar\omega S}{2}\left[1 - \exp -\left(\frac{\hbar\omega}{kT}\right)\right] = \pi\omega\left[1 - \exp -\left(\frac{\hbar\omega}{kT}\right)\right]$$
$$\times \int_0^\infty \rho(E_r+\hbar\omega)\langle E_r|Q|E_r+\hbar\omega\rangle^2 \rho(E_r) f(E_r)\, dE_r \tag{20}$$

in agreement with the result of Callen and Welton. In terms of (20), (19) becomes

$$U = \left[\frac{1}{2}\hbar\omega + \frac{\hbar\omega}{\exp(\hbar\omega/kT) - 1}\right]\left[1 - e^{-\frac{Gt}{C}}\right] + U_0 e^{-\frac{Gt}{C}} \tag{19a}$$

Equivalence of Resistance and a Noise Generator

In this section we prove the following theorem: The Johnson[4] noise plus "spontaneous" emission is entirely equivalent to all

Fig. 2. Damped electrical oscillator

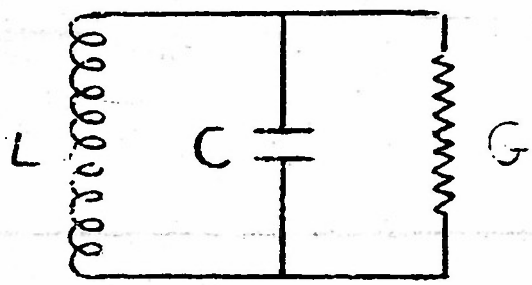
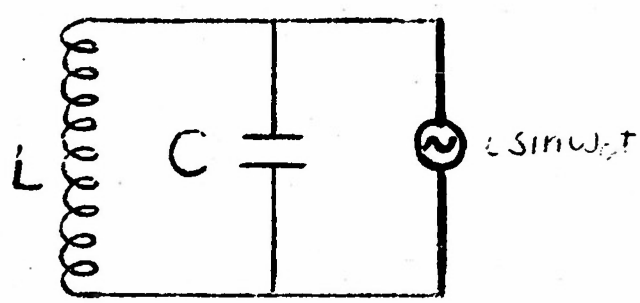


Fig. 3 Electrical oscillator and current generator



4. J. B. Johnson, Phys. Rev. 32, 97 (1928).

damping effects which the resistance has upon the oscillating circuit. The noise is seen to play a role in the damping process. To show this let us imagine that the resistance is removed, and that it is replaced by a "current generator" which has an infinite internal impedance.

It may readily be shown by comparison with the classical differential equations that the Hamiltonian of the system of Fig. 3 is

$$H = \frac{1}{2}(p^2 + \omega^2 q^2) - (qi/\sqrt{C}) \sin(\omega_g t). \tag{21}$$

It is well known that an interaction term of the form of the last term of (21) will not give a transition probability proportional to time. To obtain transitions equivalent to those of the resistance we need a current generator with a continuous spectrum in the vicinity of ω. Under these conditions the transition probability is

$$W_G = \frac{\pi\left[i^2(\omega)\right]_{Av}}{C\hbar^2}\left[\langle E_F|q|E_F + \hbar\omega\rangle^2 + \langle E_F|q|E_F - \hbar\omega\rangle^2\right], \tag{22}$$

where C is the capacitance, and the mean square value of the current over a range $d\omega$ is $\left[i^2(\omega)\right]_{Av} d\omega$. Inserting the harmonic oscillator matrix elements, (22) becomes

$$W_G = \frac{\pi\left[i^2(\omega)\right]_{Av}}{2C\hbar\omega}\left[(n+1) + n\right], \tag{23}$$

where n is the quantum number of the oscillator.

In order to compare the transition probability induced by the current generator with that induced by the resistance we insert the appropriate matrix elements into (16). Making use of (15) and (20) and rearranging terms, (16) becomes

$$W_r = \frac{G}{C}\left[\frac{(n+1)+n}{\exp(\hbar\omega/kT)-1} + n\right]. \tag{24}$$

Comparison of (23) and (24) shows that the transition probability will be the same, in so far as the first term of (24) is concerned, if

$$\left[i^2(\omega)\right]_{Av} = \frac{2G\hbar\omega}{\pi\left[\exp(\hbar\omega/kT) - 1\right]} . \qquad (25)$$

Equation (25) is the Nyquist[5] formula in the equivalent current representation, modified for quantum effects.

There is still the last term of (24). It is apparent that the last term in (24) is just the transition probability at $T = 0$, that is, the transition probability if the resistance is in its lowest state and the quantum number of the circuit is n. This is closely analogous to the spontaneous emission which atoms undergo even if the radiation fields are in their lowest states. We therefore conclude that the transitions required by (24) will be produced by a noise current generator described by (25) plus spontaneous emission, that is, plus the effect of the absorber in its lowest state.

We can get a more formal analogy with the spontaneous emission induced by the radiation modes in atoms if we imagine the second term of (24) to be equivalent to that of a current generator which can only induce downward transitions. Comparing the second terms of (23) and (24) we see that the equivalent current for such a generator is

$$\left[i^2(\omega)\right]_{Av} = 2G\hbar\omega/\pi . \qquad (26)$$

This result is formally analogous to that obtained in the treatment of spontaneous emission of radiation by Park and Epstein.[6]

<u>Mean Squared Noise Voltage and Available Power</u>

We can calculate the result of precise measurements of the mean squared noise voltage of the damped oscillator by averaging the equilibrium value of the quantity $\left[\int E \cdot d\ell\right]^2$ over the ensemble. From

5. H. Nyquist, Phys. Rev. 35, 110 (1928).

6. D. Park and H. T. Epstein, Am. J. Physics 17, 301 (1949).

(11) we obtain

$$\overline{V^2} = \left\langle \left[\int E \cdot d\bar{l} \right]^2 \right\rangle = \frac{\sum_n \langle E_{FN} | \frac{p^2}{C} | E_{FN} \rangle \exp[-(n+\frac{1}{2})\frac{\hbar\omega}{KT}]}{\sum_n \exp[-(n+\frac{1}{2})\frac{\hbar\omega}{KT}]} \qquad (27)$$

Carrying out the summations indicated in (27) we obtain

$$\overline{V^2} = \frac{1}{C}\left[\frac{1}{2}\hbar\omega + \frac{\hbar\omega}{\exp(\hbar\omega/kT)-1} \right]$$

$$= \frac{\Delta\omega}{C}\left[\frac{1}{2}\hbar\omega + \frac{\hbar\omega}{\exp(\hbar\omega/kT)-1} \right], \qquad (28)$$

where $\Delta\omega$ is the classical half-breadth. The first term of (28) is $\hbar\omega/2C$ and represents noise which would be observable even if the oscillating circuit were in its lowest state. It represents the fluctuations of the vacuum surrounding the circuit. It will now be shown that this term cannot be removed by making formal changes in the Hamiltonian which remove the zero-point energy.[7] The proof follows the discussion of the corresponding problem[8] in the quantum theory of the fields in vacuum. We introduce the auxiliary variables β and β^*, defined by

$$q = \beta + \beta^*, \quad p = -i\omega(\beta - \beta^*).$$

The Hamiltonian 8 can be written

$$H = \omega^2 \left[\beta^*\beta + \beta\beta^*\right]. \qquad (29)$$

The correspondence with the classical theory is equally good if (29) is written

$$H = 2\omega^2\beta^*\beta = \frac{1}{2}(p^2 + \omega^2 q^2) - \frac{1}{2}\hbar\omega \qquad (29A)$$

7. The author is indebted to the referee for suggesting investigation of this point.

8. W. Heitler, Quantum Theory of Radiation (Oxford University Press, London, 1944), p. 60.

The eigenvalues of the Hamiltonian (29A) do not have the zero-point energy, but the eigenfunctions of this Hamiltonian are the same as those of the Hamiltonian (8). The quantity

$$\langle E_{FN} | \frac{p^2}{C} | E_{FN} \rangle = \int \psi_{FN}^{*} \left(\frac{p^2}{C}\right) \psi_{FN} \, d\tau$$

is unchanged, and the summation (27) is also unchanged. The zero-point noise contribution is seen to be independent of the choice of zero-point energy. It is in fact due to the random interaction of the apparatus for measuring the electromotive force, with the circuit, and is related to the uncertainty principle. Equation (28) does not assert that one can observe the zero-point energy; it does assert that one can observe the zero-point fluctuations.

Another quantity which is specified in the classical discussion of resistance noise is the available power. To calculate the power which a resistance would transfer to another resistance within a specified frequency range we consider the experimental arrangement of Fig. 4. The Hamiltonian of such a system is

$$H = \frac{1}{2}(p^2 + \omega^2 q^2) + (p/\sqrt{C})\left[Q_1 + Q_2\right]. \tag{30}$$

We can deduce the expression for the rate of change of the field energy in the same manner as (18) is obtained, the result is

$$\frac{dU}{dt} = \frac{(\hbar\omega)^2}{2C}\Big\{ S_1\left[(n+1)\exp - (\hbar\omega/kT_1) - n\right] + S_2\left[(n+1)\exp - (\hbar\omega/kT_2) - n\right]\Big\} \tag{31}$$

where

$$S_1 = \frac{2\pi}{\hbar}\int_0^\infty \rho(F_{R_1} + \hbar\omega)\langle E_{R_1}|Q_1|F_{R_1} + \hbar\omega\rangle^2 \times \rho(E_{R_1}) f(E_{R_1}) dE_{R_1},$$

$$S_2 = \frac{2\pi}{\hbar}\int_0^\infty \rho(F_{R_2} + h\omega)\langle E_{R_2}|Q_2|E_{R_2} + \hbar\omega\rangle^2 \times \rho(F_{R_2}) f(E_{R_2}) dF_{R_2}.$$

The stationary value of dU/dt is obtained for a value of n which makes (31) zero. In terms of the conductances G_1 and G_2 this is

$$n = \left[\frac{G_1}{\exp(\hbar\omega/kT_1)-1} + \frac{G_2}{\exp(\hbar\omega/kT_2)-1}\right] \Big/ (G_1 + G_2). \qquad (32)$$

The average rate at which G_1 transfers energy to the system is obtained from (18) if we insert the stationary value of n as given by (32). The result is

$$\frac{G_1\hbar\omega}{C}\left[\frac{1}{\exp(\hbar\omega/kT_1)-1} - \left(\frac{G_1}{\exp(\hbar\omega/kT_1)-1} + \frac{G_2}{\exp(\hbar\omega/kT_2)-1}\right) \Big/ (G_1+G_2)\right] \qquad (33)$$

Equation (33) gives us the net power transferred to the system by G_1. Equation (33) will be a maximum if $T_2 \longrightarrow 0$ and $G_2/G_1 \longrightarrow \infty$.

$$P_{max} \longrightarrow \frac{G_1\hbar\omega}{C\left[\exp(\hbar\omega/kT_1)-1\right]} = \frac{\hbar\omega(\Delta\omega)_1}{\exp(\hbar\omega/kT_1)-1}, \qquad (34)$$

where $(\Delta\omega)_1 = G_1/C$. Equation (34) is somewhat different from the classical value because we have chosen to specify the maximum power in a way which is different from the classical one but more precise for our purposes.

Noise Measurement Experiments

Callen and Welton have given an integral for the noise of a pure resistance. They did not discuss the spectrum of the noise, and their integral contains an infinite zero point contribution.

We might measure the power spectrum of the noise by employing a filter with a flat response within the pass band and infinite rejection outside of the pass band. For simplicity we choose instead to measure the power spectrum in the vicinity of ω by connecting an LC circuit of natural frequency ω to the resistance, as in Fig. 2, and measuring

the expectation value of the square of the electromotive force. The result of such an experiment is given by Eq. (28). Although (28) was

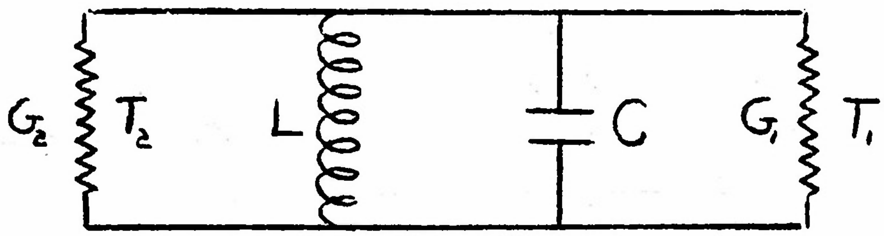


Fig. 4. Electrical oscillator coupled to conductances at different temperatures.

calculated as an average over the ensemble, the ergodic theorem guarantees that the same result will be obtained if repeated measurements are made with a single resistance. This is because the measurements do not affect the known partial specification of the state of the system. This measurement gives the noise contribution in the vicinity of ω. To obtain the noise over all frequencies we would need an infinite number of circuits of the kind discussed in this paper. The resulting zero-point contribution is therefore infinite. This is believed to be the interpretation of Callen and Welton's result.

All resistances have physical size, and there will always be a certain amount of inductance and distributed capacity. We would always have an arrangement somewhat similar to that of Fig. 2. In making the measurements we can always couple to either a single mode or at most a finite number of modes, and the zero-point noise contribution is finite.

Equation (34) shows that the maximum power which a resistance can transfer to a system tends to zero at low temperatures, while according to (28) the mean squared value of the electromotive force approaches the limit $\hbar\omega/2C$. Lawson[9] suggested noise measurements as a method

9. A. W. Lawson and E. A. Long, Phys. Rev. 70, 220 (1946).

for measuring temperature. If the noise measurements are made by measuring the transitions induced by the resistance (power measurements) there will be no zero-point contribution, according to (34). On the other hand if we measure the mean squared value of the electromotive force there will be a zero-point contribution as given by (28).

Section II

An Electrical Oscillator Damped by Radiation Resistance

The results of the previous section may be extended to include the radiation resistance, in the following way. We assume that the radiation resistance is a series element, as in Figure 1.

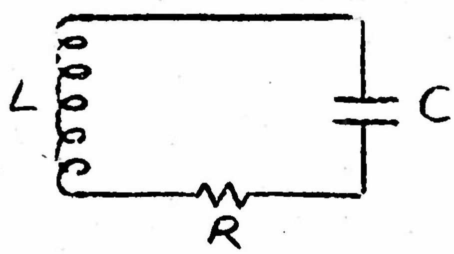


Figure 1

For the hamiltonian of Figure 1, we have

$$H = \frac{1}{2}(p^2 + \omega^2 q^2) + H_{FR} + H_e' + \sum_\lambda \frac{1}{2m_\lambda}\left(p_\lambda - \frac{e_\lambda A'}{c}\right)^2 \qquad (1)$$

The first terms of 1 are the hamiltonian of the dissipationless oscillator. The term H_{FR} is the hamiltonian of the unperturbed radiation fields, the term H_e' represents part of the hamiltonian of the electrons in the circuit which interact with the radiation fields, the last term is a summation over all the electrons and represents the interaction between electrons and the circuit and radiation fields. m_λ is the electronic mass, p_λ is the momentum, e_λ is the charge for the λ th electron. A' is the magnetic vector potential of the radiation fields. We assume that no resistance other than radiation resistance is present. The same analysis can be carried through if other resistance is present by employing essentially the procedure of the previous section.

We consider our oscillator to be weakly damped, and employ first order perturbation theory. The term in A'^2 in (1) can then be neglected. The hamiltonian (1) becomes

$$H = \frac{1}{2}(p^2 + \omega^2 q^2) + H_{FR} + H_E - \sum_\lambda \frac{e_\lambda \bar{P}_\lambda \cdot \bar{A}'}{m_\lambda c} \tag{2}$$

The term $\sum_\lambda \frac{P_\lambda^2}{2m_\lambda}$ has been included in the term H_E. This is the unperturbed hamiltonian of the electrons which interact with the radiation fields. The last term in (2) will always be a linear function of the oscillator current. In the previous section we showed that the magnetic energy is $\frac{1}{2}\omega^2 q^2$. The magnetic energy in terms of the inductance L and current I is $\frac{1}{2}LI^2$. Tho last term of 2 can be written,

$$\sum_\lambda \frac{e_\lambda \bar{P}_\lambda \cdot \bar{A}'}{m_\lambda c} = IA'F = \frac{q\omega A'F}{\sqrt{L}} \tag{3}$$

In (3), F is a function of the geometry of the system. The interaction term in (2) will cause transitions with exchange of energy between the circuit and the radiation fields through the coupling furnished by the electrons. The transition probability can be shown to be

$$W_r = \frac{2\pi}{\hbar}\Big[\rho(E_{FR} + \hbar\omega)\langle E_{FR}|A'|E_{FR}+\hbar\omega\rangle^2 \langle E_F|\frac{q\omega F}{\sqrt{L}}|E_F - \hbar\omega\rangle^2$$
$$+ \rho(E_{FR} - \hbar\omega)\langle E_{FR}|A'|E_{FR}-\hbar\omega\rangle^2 \langle E_F|\frac{q\omega F}{\sqrt{L}}|E_F+\hbar\omega\rangle^2\Big] \tag{4}$$

In (4) the symbol $\langle E_{FR}|A'|E_{FR}+\hbar\omega\rangle$ indicates the matrix element of the operator corresponding to A' between the quantum states of the radiation fields with eigenvalues E_{FR} and $E_{FR}+\hbar\omega$. $\langle E_F|\frac{q\omega F}{\sqrt{L}}|E_F+\hbar\omega\rangle$ has the corresponding meaning for the fields of the circuit. We can proceed in the manner of chapter II to derive a formula for the field energy of the circuit as a function of time. We assume that initially the circuit is in an eigenstate

but that only the temperature of the radiation field is known. We therefore need to average (4) over an ensemble of similar systems.

In (4) $\rho(E_{FR}+\hbar\omega)$ and $\rho(E_{FR}-\hbar\omega)$ are the density in energy of the quantum states of the radiation fields, and are functions of the frequency alone. To average (4) over an ensemble it is only necessary to average the squared matrix elements of the radiation fields. We imagine the radiation field to be expanded in normal modes. $A' = \sum_{\omega_i} A_{\omega_i} q_{\omega_i}$. The average of the squared matrix element $\langle E_{FR}|A'|E_{FR}+\hbar\omega\rangle^2$ is

$$\frac{\sum_i \langle E_{iFR}|A'|E_{iFR}+\hbar\omega\rangle^2 e^{-\frac{E_{iFR}}{kT}}}{\sum_i e^{-\frac{E_{iFR}}{kT}}} = \frac{\sum_0^\infty A_\omega^2 \hbar(n'+1) e^{-(n'+\frac{1}{2})\frac{\hbar\omega}{kT}}}{2\omega \sum_0^\infty e^{-(n'+\frac{1}{2})\frac{\hbar\omega}{kT}}} \tag{5}$$

(Making use of the harmonic oscillator matrix elements for the radiation fields and noting that $E_{FR}=(n'+\frac{1}{2})\hbar\omega$), (5) becomes

$$\frac{\hbar A_\omega^2}{2\omega(1-e^{-\frac{\hbar\omega}{kT}})}$$

We average the squared matrix element $\langle E_{FR}|A'|E_{FR}-\hbar\omega\rangle^2$ over the ensemble to obtain:

$$\frac{\sum_0^\infty A_\omega^2 \hbar n' e^{-(n'+\frac{1}{2})\frac{\hbar\omega}{kT}}}{2\omega\sum_0^\infty e^{-(n'+\frac{1}{2})\frac{\hbar\omega}{kT}}} = \frac{\hbar A_\omega^2 e^{-\frac{\hbar\omega}{kT}}}{2\omega(1-e^{-\frac{\hbar\omega}{kT}})} \tag{6}$$

Expression 4 becomes

$$W_r = \frac{\pi\rho(\omega)A_\omega^2}{\omega(1-e^{-\frac{\hbar\omega}{kT}})}\left[\langle E_F|\frac{g\omega F}{\sqrt{L}}|E_F-\hbar\omega\rangle^2 + \langle E_F|\frac{g\omega F}{\sqrt{L}}|E_F+\hbar\omega\rangle^2 e^{-\frac{\hbar\omega}{kT}}\right] \tag{7}$$

If we insert the harmonic oscillator matrix elements into 7 we obtain

$$W_r = \frac{\pi \rho(\omega) A_\omega^2 F^2 \hbar}{2L\left(1 - e^{-\frac{\hbar\omega}{kT}}\right)} \left[n + (n+1) e^{-\frac{\hbar\omega}{kT}} \right] \tag{8}$$

In 8 the first term represents the effect of a downward transition and the second term the effect of an upward transition. The rate of change of the field energy U, associated with the circuit is

$$\frac{dU}{dt} = \frac{\pi \rho(\omega) A_\omega^2 F^2 \hbar^2 \omega}{2L\left(1 - e^{-\frac{\hbar\omega}{kT}}\right)} \left[(n+1) e^{-\frac{\hbar\omega}{kT}} - n \right] \tag{9}$$

We can employ the relation $U = (n+\frac{1}{2})\hbar\omega$ and integrate (9) to obtain

$$U = \left[\frac{1}{2}\hbar\omega + \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1} \right] \left[1 - e^{-\frac{(\pi \rho(\omega) A_\omega^2 F^2 \hbar) t}{2L}} \right] + U_0 e^{-\frac{(\pi \rho(\omega) A_\omega^2 F^2 \hbar) t}{2L}} \tag{10}$$

For large energy (classical limit) the second term is the only significant one. Comparing (10) with the well known classical result $U = U_0 e^{-\frac{R}{L}t}$ we obtain after noting that $\rho = \frac{\omega^2}{2\pi^2 \hbar c^3}$

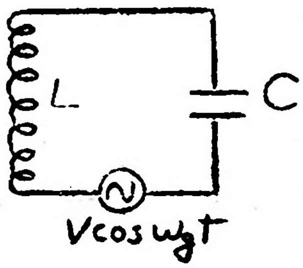
$$R = \frac{\pi \rho(\omega) A_\omega^2 F^2 \hbar}{2} = \frac{A_\omega^2 F^2 \omega^2}{4\pi c^3} \tag{11}$$

$$U = \left[\frac{1}{2}\hbar\omega + \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1} \right] \left[1 - e^{-\frac{R}{L}t} \right] + U_0 e^{-\frac{R}{L}t} \tag{12}$$

Equivalence of Radiation Resistance and a Noise Generator

In the preceding section we showed the equivalence of a noise current generator and a conductance. In this section we will show the equivalence of a noise voltage generator and a radiation resistance. We imagine first that the resistance is removed and replaced by a voltage generator.

Figure 2.



The hamiltonian of the system of figure 2 is

$$H = \frac{1}{2}(p^2 + \omega^2 q^2) + p\sqrt{C}\,V\cos\omega_0 t \tag{13}$$

It is well known that an interaction term of the form of the last term of (13) will not give a transition probability proportional to time. To obtain transitions equivalent to those of the resistance we need a voltage generator with a continuous spectrum in the vicinity of ω. Under these conditions the transition probability is

$$W_G = \frac{\pi \overline{V^2(\omega)} C}{\hbar^2} \left[\langle E_F | p | E_F + \hbar\omega \rangle^2 - \langle E_F | p | E_F - \hbar\omega \rangle^2 \right] \tag{14}$$

where C is the capacitance and the mean square value of the voltage over a range $d\omega$ is $\overline{V^2(\omega)}\,d\omega$. Inserting the harmonic oscillator matrix elements, (13) becomes

$$W_G = \frac{\pi \overline{V^2(\omega)}\, \omega C}{2\hbar} \left[(n+1) + n \right] \tag{15}$$

In order to compare the transition probability induced by the voltage generator with that induced by the radiation resistance we employ (11) to write (8) in terms of the resistance. With this substitution and the relation $\omega^2 = \frac{1}{LC}$ (8) becomes

$$W_r = RC\omega^2 \left[\frac{(n+1)+n}{e^{\frac{\hbar\omega}{kT}}-1} + n \right] \tag{16}$$

Comparison of (16) and (15) shows that the transition probability will be the same insofar as the first term of (16) is concerned if

$$\overline{V^2(\omega)} = \frac{2R\hbar\omega}{\pi\left[e^{\frac{\hbar\omega}{kT}}-1\right]} \tag{17}$$

Equation (17) is the Nyquist formula in the equivalent voltage representation, modified for quantum effects. There is still the last term in (16). This term is just the transition probability at $T = 0$, that is, the transition probability if the resistance is in its lowest state and the quantum number of the circuit is n. We conclude that for the radiation resistance the transitions required by (16) will be produced by a noise voltage generator described by (17) plus spontaneous emission, that is, plus the effect of the absorber in its lowest state.

Again we can imagine the second term of (16) to be equivalent to a voltage generator which can only induce downward transitions, comparing the second terms of (16) and (15) we see that the equivalent voltage for such a generator is

$$\overline{V^2(\omega)} = \frac{2}{\pi}\hbar\omega R \tag{18}$$

Mean Squared Noise Voltage and Available Power

We obtain the mean squared noise voltage by averaging the equilibrium value of $\left[\int \bar{E} \cdot d\bar{l}\right]^2$ over the ensemble. The result is the same and is obtained in the same way as in the previous section. The available power is also the same and is obtained in the same way.

Section III

Interaction of an Electron Stream With an Electric Circuit

In the previous sections we have shown that for a damped oscillator the results of precise measurements of the mean squared electromotive force are given by

$$\overline{V^2} = \frac{1}{C}\left[\frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1}\right] \tag{1}$$

The first term of 1 represents the effect of the vacuum fluctuations and the second term represents the effect of the thermal fluctuations. One way to observe the small voltages given by the first term of 1 is to allow the damped oscillator to interact with an electron stream and amplify the resultant electron stream noise with a radio receiver. We might argue that the first term of 1 would not be observable because it represents the noise of the oscillator in its lowest state. Since the oscillator cannot transfer energy to the electron stream when the oscillator is in its ground state, one might be tempted to conclude that there should be no zero point noise contribution. However, the circuit can gain energy from the electrons even when it is in its lowest state. This transfer of energy from electrons to the circuit takes place in a random fashion and therefore contributes noise to the electron stream. Here the electron stream and associated equipment constitute the apparatus for measuring the electromotive force of the circuit. The first term of 1 represents the effect of the random interaction of the measuring apparatus with the circuit.

It is well known that the use of electrons[10] does not in general lead to precise field measurements. It will, however, be shown here that in this case a measurement of $\overline{V^2}$ using an electron stream does lead precisely to the value given by 1.

Consider an electron stream which interacts with a damped electrical oscillator.

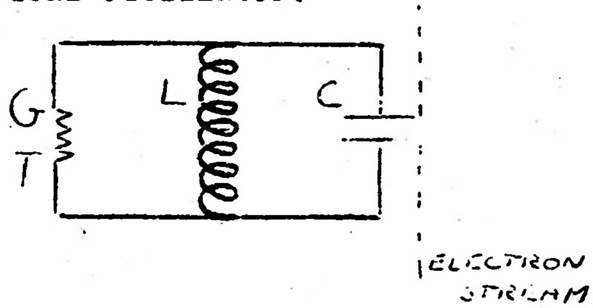


Figure 1.

The interaction can be imagined to take place by sending the stream of electrons near the condenser plates or through holes in the condenser plates. The circuit is assumed to be in thermal equilibrium with the conductance which in turn is maintained at temperature T. If the damping is small, the effect of the conductance is mainly to determine the average electromagnetic field energy of the circuit. The wavefunctions of the quantized fields of the circuit will not be significantly different from those of the undamped oscillator. Under these conditions we can employ the following hamiltonian to discuss the interaction between an electron and the fields associated with the circuit.

$$H = \frac{P^2}{2m} + \frac{1}{2}(p^2 + \omega^2 q^2) - \frac{e}{mc}\bar{A}\cdot\bar{P} \qquad (2)$$

10. W. Heitler, The Quantum Theory of Radiation (Oxford University Press, 1944), p. 78.

The variables p and q are those already defined for the circuit in section 1. P is the momentum of the electron, A is the magnetic vector potential. It is unnecessary to include terms in (2) representing the conductance because there is no direct interaction between the electron and the conductance.

We can make calculations assuming that the circuit itself is in an eigenstate of its unperturbed hamiltonian before the interaction begins, then for the period of the interaction we have for the wavefunction of the fields associated with the circuit

$$\psi = \sum_m a_m \psi_m e^{-\frac{iE_m T}{\hbar}} \quad , \text{ initially } a_n = 1, \; a_m' = 0 \quad m \neq n \tag{3}$$

we define H'_{kN} by

$$H'_{kN} = -\frac{e}{mc}\langle E_{FK}|A|E_{FN}\rangle\langle E_{eK}|P|E_{eN}\rangle \tag{4}$$

$\langle E_{FK}|A|E_{FN}\rangle$ is the matrix element of the operator corresponding to the vector potential A between the quantum states of the field with quantum numbers K and N. $\langle E_{eK}|P|E_{eN}\rangle$ is the matrix element of the operator corresponding to the electron momentum P between the two states of the electron corresponding to the K and N states of the field.

We let $\langle E_{eK}|P|E_{eN}\rangle = P_{av}$ (5)

We assume that the energy of the electron is sufficiently large so that P_{av} is almost the same as the momentum of the unperturbed electron. We again assume the vector potential A to be a product of a space and time dependent part, $\bar{A} = \bar{A}(r)\,\xi(t)$. We can now

write (4) in the form

$$H'_{KN} = -\frac{e}{mC}\,\bar{P}\cdot\bar{A}(r)\,\langle E_{FK}|q|E_{FN}\rangle \qquad (6)$$

It can be shown[11] that

$$|a_k(t)|^2 = \frac{4\,|H'_{KN}|^2 \sin^2 \tfrac{1}{2}\omega_{KN} t}{\hbar^2\,\omega_{KN}^2} \qquad (7)$$

We assume that the interaction gap is ℓ units long. From section 1 we have

$$\bar{E} = -\tfrac{1}{c}\,p\,\bar{A} \qquad p^2 = CV^2 = C(E\ell)^2 \qquad (8)$$

The total interaction time will be denoted by τ

$$\tau = \frac{\ell m}{|P|} \qquad (9)$$

Making use of (8), (9), and 6, (7) can be written

$$|a_K(\tau)|^2 = \frac{e^2}{\hbar^2 C}\left[\langle E_{FK}|q|E_{FN}\rangle^2\right]\left[\frac{\sin\tfrac{1}{2}\omega_{KN}\tau}{\tfrac{1}{2}\omega_{KN}\tau}\right]^2 \qquad (10)$$

We let $\omega_{KN}\tau = \theta$, where θ is the electron transit angle. Making use of the matrix elements for the quantum states of the field we obtain for (10)

$$|a_K(\tau)|^2 = \frac{e^2}{\hbar\omega C}\left(\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right)^2\left(\frac{n+1}{2}\right) \qquad (11)$$

$$|a_K(\tau)|^2 = \frac{e^2}{\hbar\omega C}\left(\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right)^2\left(\frac{n}{2}\right)$$

$$|a_K(\tau)|^2 = 0$$

11. W. Heitler, The Quantum Theory of Radiation, (Oxford University

Expressions 11, 12, and 13 state that the exchange of energy takes place in one quantum steps.

Suppose that M electrons interact with the circuit and that a sufficiently long time elapses between interactions so that the circuit can return to a state of equilibrium with the conductance. This will be the case if the current in the electron stream is small. Under these conditions we will have $|a_k(\tau)|^2_{k=n+1} M$ electrons lose a quantum and $|a_k(\tau)|^2_{k=n-1} M$ electrons gain a quantum. If an electron neither gains nor loses energy we can say that the electromotive force of the circuit during that interaction time was zero. If an electron gains or loses a quantum we can say that the electromotive force during that interaction time was $V = \frac{\hbar\omega}{e}$. The measured mean squared noise voltage is

$$V^2_{AVG} = \frac{\sum_0^M V_n^2}{M} = \frac{\hbar^2\omega^2}{e^2 M}\left[|a_k(\tau)|^2_{k=n+1} + |a_k(\tau)|^2_{k=n-1} \right] M = \frac{\hbar\omega}{C}\left(n+\frac{1}{2}\right)\left(\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right)^2 \qquad (14)$$

If we consider an ensemble, the value $\overline{V^2}$ must be the ensemble average. The equilibrium value of n from section 1 is

$$n_{eq} = \frac{1}{e^{\frac{\hbar\omega}{kT}} - 1} \qquad (15)$$

Inserting (15) into (14) we obtain

$$\overline{V^2_{ensemble}} = \frac{1}{C}\left[\frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1} \right]\left(\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right)^2 \qquad (16)$$

If the transit angle θ is small, (16) becomes

$$\overline{V^2} = \frac{1}{C}\left[\frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1}\right] \tag{17}$$

We see that (17) agrees with (1) and that both the thermal and the vacuum fluctuations will affect the electron stream noise. We see that the mean square random changes in velocity of the electron stream resulting from passage through the gap are the same as one would calculate classically assuming a mean squared gap voltage as given by (17).

Any circuit will have many modes. Expression (17) gives the mean squared voltage for the mode of angular frequency ω. In general the higher frequency modes will not contribute to the noise because the electron transit angle for these modes will be large. Their contribution will be reduced by the factor $\left(\frac{\sin\frac{\theta}{2}}{\frac{\theta}{2}}\right)^2$. Accordingly we will get, in the usual case, the contribution of the principal mode alone.

One might wonder why the vacuum (radiation) fields outside of the circuit do not also contribute to the noise. The radiation fields do not contribute because a free electron cannot radiate. This is because the conditions of conservation of energy and momentum cannot be simultaneously satisfied. The electron can exchange energy with the circuit because in a sense it is not free, that is, its momentum is not precisely known while it is between the condenser plates. For these reasons the circuit contributes noise while the radiation fields outside the circuit do not directly interact with the electron.

Conclusion

In this report we have examined some of the consequences of the application of field quantization to electrical circuits. The theory gives the familiar classical effects and includes in addition the noise and quantum effects. It shows clearly the role of noise in damping. The zero-point noise contribution which appeared first in the theory of Callen and Welton is shown to represent an observable effect, independent of the choice of zero-point energy. Experiments at low temperatures and high frequencies offer an opportunity to study in detail the quantum effects of a single mode of the electromagnetic field. When precise noise measurement techniques are developed it should be possible to observe directly the vacuum fluctuations in a low temperature noise experiment, presumably with an electron stream as discussed in Section III. In a subsequent report the heavily damped oscillator will be discussed. I wish to acknowledge stimulating discussions with Dr. M. H. Johnson.